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Book review

Steven Vickers, *Topology via Logic* (Cambridge University Press) 0 521 36062 5 hardback, 0 521 57651 2 paperback

Background and scope of the book. Domain theory (see [7] for a recent discussion) is a mathematical theory of computation, put forward by Scott [5] in the late 1960s and early 1970s, that provides spaces – typically called domains – over which to define computable functions. These domains can be regarded as either order-theoretic or topological structures [6]; both viewpoints are important and complementary. For instance, the order-theoretic structure provides a direct abstract notion of increase of information in the guise of a suitably complete partial order, and the topological structure suggests the notion of mapping to be considered between domains, viz. continuous function. In domain theory continuity plays the rôle of computability. For example, as advocated by Smyth [8], from a computational viewpoint open sets may be regarded as semi-decidable or computable properties. This is a standpoint for *Topology via Logic*. In this book Vickers develops a great deal of topology mainly, but not exclusively, for domain theory. Roughly speaking, the approach is to ignore the points of the spaces and to develop the theory from the algebra of open sets regarded as semi-decidable properties arising from a logic of finite observations [1]. However, the exposition also contains the relationship between this so-called localic [2] (or pointless [3]) viewpoint and the traditional spatial one via Stone-type dualities yielding the traditional order-theoretic approach to domain theory.

Topology via Logic is an unusual book that will be interesting to theoretical computer scientists and mathematicians. It will be useful to computer scientists interested in mathematical models of computation, and to domain theorists interested in the relationship between domains and program logics. Mathematicians may consult it as an introduction to locale theory (which gives an unconventional perspective of topology) with applications to computer science.

Outline of the book. Chapter 2 provides an informal discussion of a logic of affirmative assertions motivated as a logic of finite observations. This logic is technically known as (propositional) geometric logic, and its algebraic theory is that of a frame (or complete Heyting algebra).

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Frames are introduced in Chapter 3. An important mathematical example of a frame is the set of open subsets of a topological space (ordered by inclusion). An interesting example in computer science, discussed at length in the book, is the data-type of bit streams (with the prefix order).

In Chapter 4, the author goes on to treat frames as infinitary algebraic theories; that is, as sets equipped with infinitary operations subject to relations (equations and/or inequations). It is shown that the free frame over a set of generators subject to a set of relations always exists. This result is theoretically interesting and methodologically important. Theoretically, the situation is very much in contrast with the algebraic theory of complete boolean algebras, as free complete boolean algebras need not exist. Methodologically, it allows one to describe frames by generators and relations. This method is efficiently exploited by the author throughout the rest of the book.

Chapter 5 contains the first links to topology. It starts by introducing the (non-standard) notion of topological system. Roughly speaking, a topological system consists of a set of points and a frame of properties (called opens) together with a satisfaction relation, stating when a point satisfies a property, governed by the laws of the logic of finite observations. This notion allows the author to subsume both topological spaces and frames (which in this context are called locales) as topological systems, and study the relationship between them in an axiomatic framework. For example, spatial locales (locales that are topological spaces) are nothing but sober spaces. From a logical viewpoint, these are interesting because a proof of spatiality of a locale establishes the completeness of a logical system.

In Chapter 6 the following constructions on topological systems are examined: sub-systems (with particular emphasis in sub-locales), sums (or coproducts), and products. Here the method of specifying frames by generators and relations is put to use for the first time by describing the tensor product of frames, which is needed in the construction of the product of topological systems.

A first connection with traditional domain theory is established in Chapter 7. Topological systems are shown to admit an information pre-order known as the specialisation pre-order. This pre-order is discrete for Hausdorff topological spaces. However, this is typically not the case with the topological spaces of domain theory. For example, the specialisation pre-order of the data-type of bit streams coincides with the prefix order. The chapter concludes with a discussion of the closure properties of the specialisation pre-order, which are then used to motivate a fundamental notion in domain theory: the Scott topology.

In Chapter 8 the notion of compactness is introduced and studied. Examples are examined for the topological space of real numbers and the data-type of bit streams. The Hofmann–Mislove Theorem establishing a correspondence between Scott open filters of the opens of a locale and its compact saturated sets of points is proved. The notion of local compactness is also introduced and the rôle of the compact-open topology in building function spaces is discussed. These last considerations are important in topology, but diverge from the mainstream of the book (as for the spaces arising in domain theory the compact-open topology coincides with the Scott topology).

Chapter 9 is a first investigation into the spaces dealt with in domain theory. The author focusses on certain locales (called spectral algebraic) and characterises them as spaces arising from familiar order-theoretic structures in domain theory under the Scott topology. This correspondence is in the spirit of Stone's Representation Theorem for boolean algebras (also included in the text), and hence is referred to as a Stone-type duality theorem.

Chapters 10 and 11 present some constructions of domain theory used for the denotational semantics of programming languages. This is done for two well-established notions of domain, Scott domains and SFP (or bifinite) domains, for which Stone-type duality theorems are provided. The focus is on the description of lifting, products, sums, function spaces, and power domains; however, recursive domains are also mentioned briefly. A central rôle in this development is played by a method (of quasi-coprimes) for identifying whether a frame presentation yields a domain. This method relies on both spatial and localic aspects of domains.

Comments on the book. The book is well-written – the motivation, introduction, and exposition of the material is generally clear. Whilst mathematicians will find the book accessible, computer scientists will need a sound mathematical background.

Topology via Logic is especially suited as a complement to a course on domain theory. However, the book lacks a complete substantial computational application of the theory and so one will also need to consult research papers. In another vein, the book is also a valuable source of examples for a course on category theory; for instance, Chapter 4 provides good examples of adjoint functors.

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